Real cross section of the connectedness locus of the family of polynomials \((z^{2n+1} + a)^{2n+1} + b\)

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Abstract

Yeshun Sun & Yongcheng Yin [3] and H. Ishida & T. Itoh [2] presented a precise description of the real cross section of the connectedness locus of the family of bi-quadratic polynomials \(\{ (z^2 + a)^2 + b \}\). In this note, we shall give a precise description of the real cross section of the connectedness locus of the family of polynomials \(\{ P_{2n+1,b} \circ P_{2n+1,a}(z) \} = \{ (z^{2n+1} + a)^{2n+1} + b \}\), where \(a, b\) are complex numbers and \(n\) is a positive integer. Our proof is an elementary one.

Keywords: complex dynamics, real cross section, connectedness locus, polynomials, Julia set

1. Introduction and main results

Let \(\{ (P_{2n+1,b} \circ P_{2n+1,a})(z) = (z^{2n+1} + a)^{2n+1} + b \}\) be the family of polynomials with complex parameters \(a, b\), where \(n\) is a fixed positive integer. The connectedness locus of the family \(\{ P_{2n+1,b} \circ P_{2n+1,a} \}\) is the set

\[ C_{2n+1,\mathbb{C}} = \{ (a, b) \in \mathbb{C}^2 \mid \text{Julia set of } P_{2n+1,b} \circ P_{2n+1,a} \text{ is connected} \}. \]

and the real cross section of \(C_{2n+1,\mathbb{C}}\) is the set

\[ C_{2n+1,\mathbb{R}} = \{ (a, b) \in \mathbb{R}^2 \mid (a, b) \in C_{2n+1,\mathbb{C}} \}. \]

We shall prove the following

Theorem 1. \(C_{2n+1,\mathbb{R}}\) is the bounded closed region whose boundary is a simple closed curve consisting of two smooth pieces

\[ \ell_1 = \{ (a, b) \in \mathbb{R}^2 \mid a = \frac{1}{\kappa t} - t^{2n+1}, \quad b = t - \frac{1}{(\kappa t)^{2n+1}} (-t_2 \leq t \leq -t_1) \}, \]

\[ \ell_2 = \{ (a, b) \in \mathbb{R}^2 \mid a = \frac{1}{\kappa t} - t^{2n+1}, \quad b = t - \frac{1}{(\kappa t)^{2n+1}} (t_1 \leq t \leq t_2) \}, \]
where \( t_1 = \frac{1}{\sqrt{\alpha \kappa}} \), \( t_2 = \frac{\sqrt{\alpha}}{\sqrt{\kappa}} \), \( \kappa = \sqrt{2n+1} \), and \(-\alpha\) is a unique solution of the equation
\[
\sum_{j=-n}^{n} u^j = (-1)^n (2n + 1)
\]
satisfying \( u < -1 \).

2. Preliminaries

Let \( p \) be a polynomial whose degree is more than one. We denote the \( k \)-times iteration of \( p \) by \( p^k \). A critical point \( c \) of \( p \) is a zero of the derivative \( p' \) of \( p \), that is, \( p'(c) = 0 \), and a critical value of \( p \) is the image \( p(c) \) of a critical point \( c \) of \( p \). For a critical point \( c \) of \( p \), \( \{p^k(c)\}_{k=1}^{\infty} \) is a critical orbit of \( p \).

We use the following well known fact. (See [1].)

**Proposition 2.** The Julia set of \( p \) is connected if and only if all critical orbits of \( p \) are bounded.

Here we note that \( P_{2n+1,b} \circ P_{2n+1,a} \) has \( 2n + 2 \) critical points \( 0, -a^{1/(2n+1)} \). Since \( (P_{2n+1,b} \circ P_{2n+1,a})(-a^{1/(2n+1)}) = b, P_{2n+1,b} \circ P_{2n+1,a} \) has only two critical orbits. Further, these critical orbits of \( P_{2n+1,b} \circ P_{2n+1,a} \) are sequences of real numbers, if both \( a \) and \( b \) are real numbers.

3. Proof of Theorem 1.

For simplicity, we denote \( (P_{2n+1,b} \circ P_{2n+1,a})(z) = (z^{2n+1} + a)^{2n+1} + b \) by \( P(z) \). Set \( F(z) = P(z) - z \) then a fixed point of \( P \) is a solution of the equation \( F(z) = 0 \). Since
\[
P'(z) = (2n + 1)^2 z^{2n}(z^{2n+1} + a)^{2n},
\]
we have

\[ F'(z) = P'(z) - 1 = (2n + 1)^2 z^{2n}(z^{2n+1} + a)^{2n} - 1 \]
\[ = h(z)(h(z) + 2), \]

where

\[ h(z) = (2n + 1)z^n(z^{2n+1} + a)^n - 1. \]

Hereafter, we assume that both \( a \) and \( b \) be real parameters. Let denote the real part of the complex variable \( z \) by \( x \).

\[ P'(x) = 0 \] has two real solutions \( 0, -\sqrt[2n]{a} \). Let

\[ c_1 = \min(0, -\sqrt[2n]{a}) \]
\[ c_2 = \max(0, -\sqrt[2n]{a}), \]

then \( c_1 \leq c_2 \) and the equality holds when \( a = 0 \).

Since the degree of \( F(x) \) is \( (2n + 1)^2 \), \( F(x) = 0 \) has at least one real solution. Denote \( r_{\text{min}} \) and \( r_{\text{max}} \) by the least and the greatest real solution of \( F(x) = 0 \), that is, the least and the greatest real fixed point of \( P \) respectively.

We shall prove the following three lemmas.

**Lemma 3.** Both critical orbits \( \{P^k(c_1)\}_k \) and \( \{P^k(c_2)\}_k \) are bounded, that is,

\[ (a, b) \in C_{2n+1, \mathbb{R}} \]

if and only if \( F(x) = P(x) - x = 0 \) has at least two real solutions and

\[ c_1, c_2 \in [r_{\text{min}}, r_{\text{max}}] = \{x | r_{\text{min}} \leq x \leq r_{\text{max}}\}. \]

**Proof.**

First, assume that \( F(x) = P(x) - x = 0 \) has only one real solution \( x = r \). Then \( P'(r) \geq 1 \). Since \( P'(c_1) = P'(c_2) = 0 \), both \( c_1 \) and \( c_2 \) are not equal to \( r \). Hence, either \( c_1 < r \) or \( c_2 > r \). Note that \( P(x) < x \) if \( x < r \) and \( P(x) > x \) if \( x > r \). Then, if \( c_1 < r \), \( \lim_{k \to \infty} P^k(c_1) = -\infty \).

Similarly, if \( c_2 > r \), \( \lim_{k \to \infty} P^k(c_2) = \infty \).

Thus, if both critical orbits \( \{P^k(c_1)\}_k \) and \( \{P^k(c_2)\}_k \) are bounded, then \( F(x) = P(x) - x = 0 \) must have at least two real solutions.

Moreover, \( P(x) < x \) when \( x < r_{\text{min}} \) and \( P(x) > x \) when \( x > r_{\text{max}} \). Hence, if \( c_1 < r_{\text{min}} \), then \( \lim_{k \to \infty} P^k(c_1) = -\infty \). Similarly, if \( c_2 > r_{\text{max}} \), then \( \lim_{k \to \infty} P^k(c_2) = \infty \).

Henceforce, if both critical orbits \( \{P^k(c_1)\}_k \) and \( \{P^k(c_2)\}_k \) are bounded, then it holds that \( c_1, c_2 \in [r_{\text{min}}, r_{\text{max}}] \).

Conversely, assume that \( P(x) - x = 0 \) has at least two real solutions and \( c_1, c_2 \in [r_{\text{min}}, r_{\text{max}}] \). Then, \( \{P^k(c_1)\}_k, \{P^k(c_2)\}_k \subset [r_{\text{min}}, r_{\text{max}}] \), since \( P(x) \) is an increasing function.

**Lemma 4.** There exists a unique real number \( \alpha_1 < c_1 \) such that \( F'(\alpha_1) = 0 \), and there exists a unique real number \( \alpha_2 > c_2 \) such that \( F'(\alpha_2) = 0 \).
Proof.

Since \( F'(x) = h(x) \cdot (h(x) + 2) \), \( F''(x) = 2h'(x)(h(x) + 1) \), that is,

\[
F''(x) = 2n(2n + 1)^2 x^{2n-1}(x^{2n+1} + a)^{2n-1}(2(n+1)x^{2n+1} + a).
\]

Set \( c^* = \frac{-2n\sqrt{a}}{2n\sqrt{2(n+1)}} \), then \( c_1 \leq c^* \leq c_2 \). Moreover, \( F''(x) < 0 \) if \( x < c_1 \) and \( F''(x) > 0 \) if \( x > c_2 \). Since \( F'(c_1) = F'(c_2) = -1 \), it is easy to verify that Lemma 4 holds.

Lemma 5. \( F(x) = 0 \) has at least two real solutions and \( c_1, c_2 \in [r_{\min}, r_{\max}] \) if and only if \( F(a_1) \geq 0 \) and \( F(a_2) \leq 0 \).

Proof.

Note that \( F(x) \) is decreasing when \( c_2 < x < a_2 \), increasing when \( \alpha_2 < x \) and \( \lim_{x \to \infty} F(x) = \infty \). Further, \( F(x) \) is increasing when \( x < \alpha_1 \), decreasing when \( \alpha_1 < x < c_1 \), and \( \lim_{x \to -\infty} F(x) = -\infty \). So, it is easy to verify that Lemma 5 holds.

Proof of Theorem 1.

Recall that \( F'(x) = h(x) \cdot (h(x) + 2) \), where

\[
h(x) = (2n + 1)x^n(x^{2n+1} + a)^n - 1.
\]

Then

\[
h'(x) = n(2n + 1)x^{n-1}(x^{2n+1} + a)^n-1(2(n+1)x^{2n+1} + a)
\]

Since \( h(c_1) = h(c_2) = -1 \), \( \alpha_1, \alpha_2 \) are determined by the relations \( h(\alpha_1) = h(\alpha_2) = 0 \) and \( \alpha_1 < c_1 \leq c_2 < \alpha_2 \).

Therefore, for any \( a \), there is a unique \( t < c_1 \) such that

\[
h(t) = (2n + 1)t^n(t^{2n+1} + a)^n - 1 = 0,
\]

that is,

\[
t(t^{2n+1} + a) = \frac{1}{\sqrt{2n + 1}}. \tag{1}
\]

For the value \( t \),

\[
F(t) = (t^{2n+1} + a)^{2n+1} + b - t = \left( \frac{1}{t\sqrt{2n + 1}} \right)^{2n+1} + b - t \geq 0
\]

if and only if \( F(\alpha_1) \geq 0 \).

The relation (1) determines a smooth curve

\[
a = \frac{1}{kt} - t^{2n+1} \quad (t < 0),
\]

where \( \kappa = \sqrt{2n + 1} \). Hence, \( F(\alpha_1) = 0 \) if and only if

\[
b = t - \frac{1}{(kt)^{2n+1}}.
\]
From these two relations with respect to $a$ and $b$, we determine the boundary curve $\ell_1$ of $C_{2n+1,\mathbb{R}}$.

Let

$$\xi = a - b = \frac{1}{kt} - t - t^{2n+1} + \frac{1}{(kt)^{2n+1}} \quad (t < 0),$$

$$\eta = a + b = \frac{1}{kt} + t - t^{2n+1} - \frac{1}{(kt)^{2n+1}} \quad (t < 0),$$

then $\eta$ is a single valued function $\eta(\xi)$ of $\xi (-\infty < \xi < \infty)$ and satisfies

$$\xi \left( \frac{1}{kt} \right) = -\xi(t), \quad \eta \left( \frac{1}{kt} \right) = \eta(t).$$

Further,

$$\frac{d\eta}{d\xi} = \frac{-1 + \kappa^2t^{2n} + \frac{1}{\kappa^{2n+1}t^{2n+2}}}{\frac{1}{kt^2} - 1 - \kappa^2t^{2n} - \frac{1}{\kappa^{2n+1}t^{2n+2}}} = \frac{(kt^2)^n - 1}{(kt^2)^n + 1} \quad (t < 0),$$

$$\frac{d^2\eta}{d\xi^2} = \frac{-4n\kappa^{2n+1}t^{4n+1}}{(k^n t^{2n} + 1)^2(k^{2n+1}t^{2n+2} + 1)} \quad (t < 0).$$

Hence, $\eta(\xi)$ is convex and has a unique minimal value

$$\frac{2(-\kappa^n + 1)}{\kappa^n \sqrt{k}} = -\frac{4n}{(2n + 1) \sqrt{2n + 1}}$$

at $\xi = 0$, which corresponds to $t = -1/\sqrt{k}$. Clearly, $\lim_{\xi \to \pm \infty} \eta = +\infty$. Hence, in $\xi\eta$-plane, $\eta = \eta(\xi)$ transverses $\eta$-axis twice. By relation (3), we know that $\eta = 0$ has only two solutions in $t < 0$, whose product is $1/\kappa$. Since $\eta < 0$ when $t = -1$, one of these solutions is less than $-1$ and another is between $-1/\kappa$ and $0$.

The equation $\eta = 0$ of $t$ implies

$$\kappa^{2n+1}t^{4n+2} + 1 = \kappa^{2n}(kt^2 + 1).$$

Let $kt^2 = -u$ then we have

$$\sum_{j=-n}^{n} u^j = (-1)^n(2n + 1) \quad (4)$$

and this equation has a unique solution satisfying $u < -1$. Denote the solution by $\alpha$, then $-1/\alpha$ is another solution of (4). Therefore, $\sqrt{k\alpha}$, $\sqrt{\alpha/k}$ are two solutions of $\eta = 0$.

Similarly, by the condition $F(\alpha_2) = 0$, we have

$$a = \frac{1}{\kappa s} - s^{2n+1} \quad (s > 0),$$

and

$$b = s - \frac{1}{(\kappa s)^{2n+1}} \quad (s > 0).$$

Set $t = -s$, then we get

$$-a = \frac{1}{kt} - t^{2n+1} \quad (t < 0).$$
and

\[-b = t - \frac{1}{(kt)^{2n+1}} \quad (t < 0).\]

Hence, another boundary curve \( \ell_2 \) of \( C_{2n+1, R} \) is symmetric to \( \ell_1 \) with respect to the origin in the \( ab \)-plane. \( \blacksquare \)

**References**


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多項式族 \((z^{2n+1} + a)^{2n+1} + b\) の連結性集合の実断面

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要 旨

関数族 \((z^{2n+1} + a)^{2n+1} + b\) の連結性集合の実断面を表わす式を決定した。

キーワード：複素力学系, 実断面, 連結性集合, 多項式, Julia 集合